

Homework 2

Math 121A, Fall 2025.

Due: Wednesday, Sep 10, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problems are taken from Boas. It might look like a lot, but most of these problems are much easier than last week's.

Problem 1. Find the interval of convergence of each of the following power series. Be sure to investigate the endpoints of the interval in each case.

- (a) $\sum_{n=0}^{\infty} (2x)^n$.
- (b) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$.
- (c) $\sum_{n=1}^{\infty} n(-2x)^n$.
- (d) $\sum_{n=5}^{\infty} \frac{x^{3n}}{n}$.

Problem 2. The following series are not power series, but you can use a change of variables to transform them into a power series. Do this to figure out where they converge.

- (a) $\sum_{n=0}^{\infty} \frac{n!(-1)^n}{x^n}$.
- (b) $\sum_{n=0}^{\infty} (x^2 + 1)^{2n}$.

Problem 3. Section 1.13

- (a) Problem 9
- (b) Problem 12
- (c) Problem 23

Problem 4. Section 1.14 (Note: instead of using the test in Boas's book, I recommend using Taylor's theorem with remainder.)

- (a) Problem 3
- (b) Problem 5
- (c) Problem 8

Problem 5. Section 1.15

- (a) Problem 2
- (b) Problem 6
- (c) Problem 8

- (d) Problem 9
- (e) Problem 11
- (f) Problem 28
- (g) Problem 29 (up to order $o(\theta^6)$ is fine).

Problem 6. (*Change of coordinates for Taylor series*)

- (a) Consider the exponential function, which we described by its power series centered at the origin $0 \in \mathbb{R}$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_n x^n$$

Compute a_0, a_1, a_2 .

- (b) Now, suppose we expanded the exponential function as a power series centered at the point $1 \in \mathbb{R}$:

$$e^x = \sum_{n=0}^{\infty} \tilde{a}_n (x-1)^n.$$

Compute $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2$.

Problem 7. (*A warning about Taylor expansions*) We already know that a Taylor expansion of a function might not be possible if the function is not smooth enough (e.g. $|x|$ has a discontinuity in its derivative). Time for another possibility.

- (a) Consider the function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\psi(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Draw the graph of this function.

- (b) Show that this function has derivative $\psi'(0) = 0$ by using the limit definition of the derivative:

$$\psi'(0) = \lim_{h \rightarrow 0} \frac{\psi(h) - \psi(0)}{h}.$$

Hint: when $h > 0$, we have $\frac{1}{h^2} \leq e^{1/h}$, which implies that $\frac{1}{h} = h \frac{1}{h^2} \leq h e^{1/h}$.

- (c) Essentially by repeating the above calculation, it is not too hard to show that the k^{th} derivative is $\psi^{(k)}(0) = 0$. What is the Taylor expansion of ψ about 0?
- (d) Let f be a function with all derivatives (aka an infinitely differentiable function). Is f equal to its Taylor series?

Problem 8. Use Taylor series to approximate

$$\int_0^1 \frac{\sin x}{x} dx$$

within 1% of its true value.